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ON THE ARITHMETICO-GEOMETRICAL MEAN.

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GAUSS, in his Memoir entitled "Determinatio Attractionis", &c., Werke, Bd. 3, p. 333, has given a method for finding the value of the elliptic integral

$$\int \frac{d\theta}{\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)}},$$

which can be developed in the following manner:

Let m and n be any two real and positive quantities, then if m_1 and n_1 are their Arithmetical and Geometrical means so that

$$m_1 = \frac{m+n}{2};\quad n_1 = \sqrt{mn};$$

$$m_2 = \frac{m_1+n_1}{2};\quad n_2 = \sqrt{m_1n_1};\ \&c.,$$

the successive values of m and n will very rapidly converge to a definite limit μ which for simplicity Gauss calls the Arithmetico-Geometrical Mean.

As an example take

$$\begin{array}{ll}
m\ \ = 100, & n\ \ = 1; \\
m_1 = 50.5, & n_1 = 10; \\
m_2 = 30.25, & n_2 = 22.47221; \\
m_3 = 26.36110, & n_3 = 26.07267; \\
m_4 = 26.21689, & n_4 = 26.21649; \\
m_5 = 26.21669, & n_5 = 26.21669.
\end{array}$$

This calculation is made quite readily for each m is the arithmetical mean of the preceding m and n; also the logarithm of each n is the arithmetical mean of the logarithms of the preceding m and n; which logarithm will be used in computing the next n, &c.

I. To prove that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)}} = \frac{1}{\mu},$$

suppose the variable θ to be represented as a function of another variable θ_1 so that

$$\sin\theta = \frac{2m\sin\theta_1}{(m+n)\cos^2\theta_1+2m\sin^2\theta_1}. \tag{1}$$

We know that this is always possible, for the value of θ_1 which makes the right hand member of this equation a maximum or minimum is given by the equation $\cos\theta_1=0$; the other equation, $(m-n)\sin^2\theta_1=m+n$, is plainly inadmissible. Putting therefore

$$\theta_1 = \tfrac{1}{2}\pi \text{ or } \tfrac{3}{2}\pi$$

we find that this expression always lies between $+1$ and -1; θ and θ_1 will always have the same sign and the limits will be the same from 0 to $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$ or 2π.

Substituting $1-\sin^2\theta_1$ for $\cos^2\theta_1$ in (1) we have

$$\left.\begin{aligned}\sin\theta &= \frac{2m\sin\theta_1}{m+n+(m-n)\sin^2\theta_1},\\ \text{or } (m-n)\sin\theta\sin^2\theta_1 - 2m\sin\theta_1 &+ (m+n)\sin\theta = 0,\end{aligned}\right\} \tag{2}$$

and solving this equation for θ_1 we have

$$(m-n)\sin\theta\sin\theta_1 = m\pm\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)},$$

or

$$\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)} = m-(m-n)\sin\theta\sin\theta_1. \tag{3}$$

Differentiating equation (2) under the following form,

$$\frac{\sin\theta}{\sin\theta_1} = \frac{m+n+(m-n)\sin^2\theta_1}{2m},$$

we have

$$\sin\theta\cos\theta_1 d\theta_1 - \sin\theta_1\cos\theta d\theta = \frac{m-n}{m}\sin\theta_1\cos\theta_1\sin^2\theta d\theta_1,$$

or

$$\frac{d\theta}{m-(m-n)\sin\theta\sin\theta_1} = \frac{d\theta_1}{m\cot\theta\tan\theta_1} = \text{from (3) } \frac{d\theta}{\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)}}. \tag{4}$$

Expressing the denominator $m\cot\theta\tan\theta_1$ as a function of m_1, n_1 and θ_1 we have

$$m^2\cot^2\theta\tan^2\theta_1 = m^2\tan^2\theta_1\left(\frac{1}{\sin^2\theta}-1\right)$$

from (2),

$$= m^2\tan^2\theta_1\left[\frac{(m+n)^2}{4m^2\sin^2\theta_1}+\frac{(m-n)^2}{4m^2}\sin^2\theta_1+\frac{m^2-n^2}{2m^2}-1\right].$$

On substituting $(m+n)^2 = 4m_1^2$, $(m-n)^2 = 4(m_1^2-n_1^2)$ and $\frac{1}{2}(m^2+n^2) = 2m_1^2-n_1^2$ this equation becomes

$$\begin{aligned}&= \frac{1}{\cos^2\theta_1}\Big[m_1^2(\cos^2\theta_1+\sin^2\theta_1)^2+(m_1^2-n_1^2)\sin^4\theta_1\\ &\qquad\qquad -(2m_1^2-n_1^2)\sin^2\theta_1(\cos^2\theta_1+\sin^2\theta_1)\Big]\\ &= m_1^2\cos^2\theta_1+n_1^2\sin^2\theta_1.\end{aligned} \tag{5}$$

Equation (4) becomes

$$\frac{d\theta}{\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)}} = \frac{d\theta_1}{\sqrt{(m_1^2\cos^2\theta_1+n_1^2\sin^2\theta_1)}}, \tag{6}$$

and therefore

$$\int_0^{2\pi}\frac{d\theta}{2\pi\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)}} = \int_0^{2\pi}\frac{d\theta_1}{2\pi\sqrt{(m_1^2\cos^2\theta_1+n_1^2\sin^2\theta_1)}};$$

and proceeding in the same way with θ_2, m_2, n_2,&c., we find it to be ultimately equal to

$$\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{(\mu^2\cos^2\theta + \mu^2\sin^2\theta)}} = \frac{1}{\mu}. \tag{7}$$

II. From this value many more interesting integrals may be found. Resuming equations (3) and (5) we have

$$\sqrt{(m^2\cos^2\theta + n^2\sin^2\theta)} = m - (m - n)\sin\theta\sin\theta_1,$$
$$\sqrt{(m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1)} = m\cot\theta\tan\theta_1;$$

whence we find

$$\begin{aligned}
&\sin\theta\sin\theta_1\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)} - \cos\theta\cos\theta_1\sqrt{(m_1^2\cos^2\theta_1+n_1^2\sin^2\theta_1)}\\
&= m\sin\theta\sin\theta_1 - (m-n)\sin^2\theta\sin^2\theta_1 - m\cos\theta\cot\theta\sin\theta_1\\
&= m\sin\theta\sin\theta_1(1-\cot^2\theta) - (m-n)\sin^2\theta\sin^2\theta_1\\
&= -m\frac{\sin\theta_1}{\sin\theta}(\cos^2\theta - \sin^2\theta) - (m-n)\sin^2\theta\sin^2\theta_1\\
&= -\tfrac{1}{2}(\cos^2\theta - \sin^2\theta)\left[m+n+(m-n)\sin^2\theta_1\right] - (m-n)\sin^2\theta\sin^2\theta_1, \text{ by (2)},\\
&= -\tfrac{1}{2}(m+n)(\cos^2\theta - \sin^2\theta) - \tfrac{1}{2}(m-n)(1-2\sin^2\theta)\sin^2\theta_1 - (m-n)\sin^2\theta\sin^2\theta_1\\
&= -\tfrac{1}{2}(m+n)(\cos^2\theta - \sin^2\theta) - \tfrac{1}{2}(m-n)\sin^2\theta_1; \text{ therefore}
\end{aligned}$$

$$\begin{aligned}
&\sin\theta\sin\theta_1\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)} + \tfrac{1}{2}(m+n)(\cos^2\theta - \sin^2\theta)\\
&\qquad = \cos\theta\cos\theta_1\sqrt{(m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1} - \tfrac{1}{2}(m-n)\sin^2\theta_1.
\end{aligned}$$

Multiplying this equation by each member of (6) we have

$$\frac{(m+n)(\cos^2\theta - \sin^2\theta)d\theta}{2\sqrt{(m^2\cos^2\theta+n^2\sin^2\theta)}} = -\frac{(m-n)\sin^2\theta_1\,d\theta_1}{2\sqrt{(m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1)}} + d.\sin\theta_1\cos\theta. \tag{8}$$

Multiplying this equation by $(m-n)\div\pi$ and substituting $(m-n)^2 = 4(m_1^2 - n_1^2)$, $\sin^2\theta_1 = \frac{1}{2} - \frac{1}{2}(\cos^2\theta_1 - \sin^2\theta_1)$ and integrating from 0 to 2π we have

$$\begin{aligned}
(m^2-n^2)\int_0^{2\pi}\frac{(\cos^2\theta - \sin^2\theta)d\theta}{2\pi\sqrt{(m^2\cos^2\theta + n^2\sin^2\theta)}} &= -\frac{2(m_1^2 - n_1^2)}{\mu}\\
&+ 2(m_1^2 - n_1^2)\int_0^{2\pi}\frac{(\cos^2\theta_1 - \sin^2\theta_1)d\theta_1}{2\pi\sqrt{(m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1)}};
\end{aligned}$$

and by successive substitutions in the same way it becomes

$$\begin{aligned}
\int_0^{2\pi}\frac{(\cos^2\theta - \sin^2\theta)d\theta}{2\pi(m^2\cos^2\theta + n^2\sin^2\theta)} &= -\frac{2(m_1^2 - n_1^2) + 4(m_2^2 - n_2^2) + 8(m_3^2 - n_3^2) + \&\text{c.}}{(m^2-n^2)\mu}\\
&= -\frac{\nu}{\mu}. \qquad (9)
\end{aligned}$$

If we now put $\frac{1}{4}\sqrt{(m^2 - n^2)} = \lambda$; $\frac{1}{4}\sqrt{(m_1^2 - n_1^2)} = \lambda_1$; $\frac{1}{4}\sqrt{(m_2^2 - n_2^2)} = \lambda_2$, &c., we find that

$$\lambda_1 = \frac{\lambda^2}{m_1};\ \lambda_2 = \frac{\lambda_1^2}{m_2}\ \&\text{c.}; \qquad \nu = \frac{2\lambda_1^2 + 4\lambda_2^2 + 8\lambda_3^2 + \&\text{c.}}{\lambda^2}.$$

If we now put $\omega = \surd(m^2\cos^2\theta + n^2\sin^2\theta)$ we find, from (7) and (9),

$$\int_0^{2\pi} \frac{\cos^2\theta d\theta}{2\pi\omega} = \frac{1-\nu}{2\mu}; \qquad \int_0^{2\pi} \frac{\sin^2\theta d\theta}{2\pi\omega} = \frac{1+\nu}{2\mu}. \tag{10}$$

If we put

$$P = \int_0^{2\pi} \frac{\cos^2\theta d\theta}{2\pi\omega^3}; \qquad Q = \int_0^{2\pi} \frac{\sin^2\theta d\theta}{2\pi\omega^3},$$

we can find their values in the following manner:

$$d.\frac{\sin\theta\cos\theta}{\pi\omega} = \frac{(\cos^2\theta - \sin^2\theta)d\theta}{\pi\omega} + \frac{(m^2 - n^2)\sin^2\theta\cos^2\theta d\theta}{\pi\omega^3}$$

$$= \frac{(\cos^2\theta - \sin^2\theta)d\theta}{\pi\omega} + \left[\frac{(m^2\cos^2\theta - n^2\sin^2\theta)(\cos^2\theta + \sin^2\theta)}{2\pi\omega^3} - \frac{(m^2\cos^2\theta + n^2\sin^2\theta)(\cos^2\theta - \sin^2\theta)}{2\pi\omega^3}\right] d\theta$$

$$= \frac{(\cos^2\theta - \sin^2\theta)d\theta}{2\pi\omega} + \left[\frac{m^2\cos^2\theta - n^2\cos^2\theta}{2\pi\omega^3}\right] d\theta. \tag{11}$$

Integrating this equation between the limits 0 and 2π, we have

$$-\frac{\nu}{\mu} + m^2P - n^2Q = 0, \text{ and from (7) } -\frac{1}{\mu} + m^2P + n^2Q = 0;$$

$$\therefore \quad P = \frac{1+\nu}{2m^2\mu}; \qquad Q = \frac{1-\nu}{2n^2\mu}; \text{ and } \int_0^{2\pi} \frac{d\theta}{2\pi\omega^3} = P + Q. \tag{12}$$

It is evident from equations (8) and (11) that the integrals from 0 to $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$ and 2π will be proportional to these values , for $\sin\theta_1\cos\theta$ and $\sin\theta\cos\theta$ will vanish at both limits.

NOTE.—The indefinite integral

$$\int \frac{d\zeta^{*}}{\surd(m^2\cos^2\zeta + n^2\sin^2\zeta)} = \frac{\theta}{\mu} \text{ from (7).}$$

Multiplying (8) by $2(m-n)$ we find

$$\int \frac{(\cos^2\zeta - \sin^2\zeta)d\zeta}{\surd(m^2\cos^2\zeta + n^2\sin^2\zeta)} = -\frac{\nu\theta}{\mu} + \frac{\lambda_1\cos\zeta\sin\zeta_1 + 2\lambda_2\cos\zeta_1\sin\zeta_2 + \&c.}{\lambda_2},$$

where θ is the limiting value of ζ if ζ_2 is determined from ζ_1, m_1 and n_1 in the same manner that ζ_1 is determined from m, n, ζ, &c.

Collecting all these results and putting, for the sake of brevity,

$$\omega = \surd(m^2\cos^2\zeta + n^2\sin^2\zeta)$$

we have

$$\int_0^{2\pi} \frac{d\zeta}{2\pi\omega} = \frac{1}{\mu},$$

$$\int_0^{2\pi} \frac{(\cos^2\zeta - \sin^2\zeta)d\zeta}{2\pi\omega} = -\frac{\nu}{\mu},$$

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[*] For want of sorts, ζ is here, and through the succeeding part of this Art., used for θ.

$$\int_0^{2\pi}\frac{\cos^2\zeta d\zeta}{2\pi\omega}=\frac{1-\nu}{2\mu};\qquad \int_0^{2\pi}\frac{\sin^2\zeta d\zeta}{2\pi\omega}=\frac{1+\nu}{2\mu};$$

$$\int_0^{2\pi}\frac{\cos^2\zeta d\zeta}{2\pi\omega^3}=\frac{1+\nu}{2m^2\mu};\qquad \int_0^{2\pi}\frac{\sin^2\zeta d\zeta}{2\pi\omega^3}=\frac{1-\nu}{2n^2\mu};$$

$$\int_0^{2\pi}\frac{d\zeta}{2\pi\omega^3}=\frac{1+\nu}{2m^2\mu}+\frac{1-\nu}{2n^2\mu}=\frac{m^2+n^2-(m^2-n^2)\nu}{2m^2n^2\nu};$$

and the integrals from 0 to $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$ and 2π are always proportional to them.

The indefinite integrals

$$\int\frac{d\zeta}{\omega}=\frac{\theta}{\mu};$$

$$\int\frac{(\cos^2\zeta-\sin^2\zeta)d\zeta}{\omega}=-\frac{\nu\theta}{\mu}+\frac{\lambda_1\cos\zeta\sin\zeta_1+\lambda_2\cos\zeta\sin\zeta_2+\&c.}{\lambda^2}=-\frac{\nu\theta}{\mu}+\Lambda;$$

$$\int\frac{\cos^2\zeta d\zeta}{\omega}=\frac{(1-\nu)\theta}{2\mu}+\frac{\Lambda}{2};\quad \int\frac{\sin^2\zeta d\zeta}{\omega}=\frac{(1+\nu)\theta}{\mu}-\frac{\Lambda}{2};$$

$$\int\frac{\cos^2\zeta d\zeta}{\omega^3}=\frac{(1+\nu)\theta+\left(\frac{\sin 2\zeta}{\omega}-\Lambda\right)\mu}{2m^2\mu};\quad \int\frac{\sin^2\zeta d\zeta}{\omega^3}=\frac{(1-\nu)\theta-\left(\frac{\sin 2\zeta}{\omega}-\Lambda\right)\mu}{2n^2\mu};$$

$$\int\frac{d\zeta}{\omega^3}=\frac{\left(m^2+n^2-(m^2-n^2)\nu\right)\theta-(m^2-n^2)\left(\frac{\sin 2\zeta}{\omega}-\Lambda\right)\mu}{2m^2n^2\nu};$$

where ν, Λ, &c., have the values given above.

THE SUMMATION OF SERIES WHOSE COEFFICIENTS FORM AN ARITHMETICAL PROGRESSION OF ANY ORDER.

BY PROF. J. SCHEFFER, MERCERSBERG, PA.

LET the series be represented by $ax+bx^2+cx^3+dx^4+\ldots$ whose sum we denote by S. Putting $x=y\div(1+y)$, we have

$$S=a\left(\frac{y}{1+y}\right)+b\left(\frac{y}{1+y}\right)^2+c\left(\frac{y}{1+y}\right)^3+d\left(\frac{y}{1+y}\right)^4+\ldots\ldots$$

Developing each of these fractions by the Binomial Theorem, we have

$$S=ay-(a-b)y^2+(a-2b+c)y^3-(a-3b+3c-d)y^4+(a-4b+6c-4d+e)y^5-\ldots$$

We see that the coefficients of this series are the successive finite differences, which we shall denote by Δa, Δ^2a, Δ^3a, ..., consequently

$$S=ay+\Delta a.y^2+\Delta^2a.y^3+\Delta^3a.y^4+\ldots \text{ or}$$

$$S=a\left(\frac{x}{1-x}\right)+\Delta a\left(\frac{x}{1-x}\right)^2+\Delta^2a\left(\frac{x}{1-x}\right)^3+\Delta^3a\left(\frac{x}{1-x}\right)^4\ldots$$